Descriptive Set Theory
Lecture 9

Examples of nowhere dense sets.

- The Cantor set $C \subseteq [0,1]$ is nowhere dense because it's closed and has no interior.
- In $\mathbb{N}^{\mathbb{N}}$, for any finitely-branching tree T, the set $[T]$ is nowhere dense because it's closed and has no interior.

Meager sets. Note that although nowhere dense sets form an ideal, they are not closed under ctbl unions. Indeed, $\mathbb{Q}$ is a ctbl union of singletons, each of which is nowhere dense because $\mathbb{R}$ is perfect, but $\mathbb{Q}$ itself is dense in $\mathbb{R}$ (hence not nowhere dense).

Then we call a set $A \subseteq X$ in a top. space X *meager* if it is a ctbl union of nowhere dense sets. A is *comeager* if its complement is meager. By def, meager sets form a σ-ideal, i.e. their class is closed under ctbl unions.

Upgrade property. Let X be a top. space and $A \subseteq X$.

(a) A is meager $\iff$ A is contained in a meager F_σ set; in fact $A \subseteq \bigcup_n C_n$, where C_n is closed and has empty interior.

(b) A is comeager $\iff$ A contains a comeager G_δ set; in fact, $A \supseteq \bigcap_n U_n$, where U_n is dense open.

Examples
- In $\mathbb{N}^{\mathbb{N}}$, any K_σ set is meager because each compact set is nowhere dense.
- $\mathbb{Q}$ is meager in $\mathbb{R}$, also in itself.

Relativization to a subset. Let X be a top. space and $Y \subseteq X$. Some top. concepts are absolute between X and Y, i.e. a property P s.t. for a subset $A \subseteq Y$,

$$A \text{ has } P \text{ within } Y \iff A \text{ has } P \text{ within } X.$$

Such absolute properties include: connectedness, compactness, being nonempty, metrizability.

Prop. Let X be a top. space and $Y \subseteq X$ and $A \subseteq Y$.

(a) If A is nowhere dense (resp meager) in Y then A is nowhere dense (resp meager) in X.

(b) If Y is open, then the converse of (a) also holds.

Baire spaces. Meager sets, despite their name, may be large sets in some spaces, e.g. $\mathbb{Q}$ is meager but relative to itself it's the whole space. In other words, the σ-ideal $\mathrm{MGR}(X)$ of meager sets may be $\mathcal{P}(X)$, i.e. it trivializes. The following isolates spaces where this doesn't happen (even locally).

Def. We call a top. space **Baire** if no nonempty open set is meager.

Prop. For a top. space X, TFAE.

(1) X is Baire, i.e. every nonempty open set is nonmeager.

(2) Every comeager set is dense (in particular, nonempty if $X \neq \emptyset$).

(3) Ctbl intersections of open dense sets are dense.

Proof. (2) $\Rightarrow$ (3) as a special case, and (3) $\Rightarrow$ (2) by the upgrade property.

(1) $\Rightarrow$ (2). Complements of comeager sets are meager, hence cannot contain a nonempty open set.

(2) $\Rightarrow$ (1). For a meager open set U, its complement is

comeager, hence dense, so $U=\emptyset$. $\square$

Obs. In Baire spaces, comeager $\Leftrightarrow$ containing dense G_δ.

Obs. In Baire spaces, nonempty open sets are themselves Baire spaces.

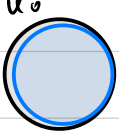
Useful fact. In any 2nd ctbl top. space X, $\exists$ comeager G_δ subset Y that is 0-dim.

Proof. Let (U_n) be a ctbl basis and put $Y := X \setminus \bigcup_n \partial U_n$. $\square$

Baire category theorem. Complete metric spaces are Baire. Also, locally compact Hausdorff spaces are Baire.

Proof. Let (X,d) be a complete metric space (the loc. compact case is left as homework). Let V_n be open dense sets and we show that $\bigcap_n V_n$ is dense. Fix a nonempty open set U_0 and show that $\exists\, x \in U_0 \cap \bigcap_n V_n$.

U_0



$U_0 \cap V_0$

We play the following game:

Player 1:	U_0	U_1	U_2	...
Player 2:	$U_0\cap V_0$	$U_1 \cap V_1$	$U_2 \cap V_2$	

where $\overline{U_{n+1}} \subseteq U_n \cap V_n$, $U_n \neq \emptyset$, $\operatorname{diam}(U_{n+1}) \leq \frac{1}{n+1}$.

Thus, the $\overline{U}_n$ are decreasing & of vanishing diameter so $\bigcap_n \overline{U}_n = \{x\}$, by completeness. But

$$\bigcap_n \overline{U}_n = \bigcap_n U_n \subseteq U_0 \cap \bigcap_n V_n,$$ so the latter is nonempty. $\square$

The proof can be carried out more generally in the so-called Choquet spaces, namely, a space X where in the Choquet game, Player 1 has a winning strategy. The Choquet game is:

Player 1:	U_0		U_1		$\dots$
Player 2:		V_0		V_1	

where $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq U_2 \supseteq \dots$ Player 2 wins if $\bigcap_n U_n (= \bigcap_n V_n) \neq \emptyset$. Our proof above shows that Choquet spaces are Baire. It also shows that complete metric spaces are Choquet.

Cor. In Polish spaces, dense meager sets are not G_δ, hence not Polish in the relative topology. E.g. $\mathbb{Q}$ is not Polish.

We say that a property P in a Baire space X holds **generically** if P holds for comeagerly many points in X, or we say that a **generic point** has P.
This way of thinking compresses some number of quantifiers into 1, and provides a method of proof of existence. E.g. one can show that a generic continuous function on $[0,1]$ is nowhere differentiable, hence there is a nowhere differentiable function, while constructing such a function explicitly is more involved. This started with Cantor, who showed that transcendental numbers exist by showing that all but ctbly many reals are transcendental. An explicit construction of such a number was given by Liouville earlier, but the proof was again not easy. This is what popularized set theory.

Regularity properties of subsets of Polish spaces

Games and determinacy. We'll see that infinite games and whether one of the two players has a winning strategy are tightly connected to the more analytic regularity properties of subsets of Polish spaces, e.g. measurability, the perfect set property, Baire measurability.